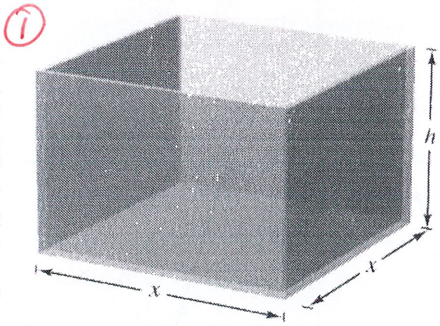
A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?

Step 1



Open box with square base:
$S = x^2 + 4xh = 108$

What are we needing to maximize?

Step 2 Volume $= V(x,h) = x^2 h$

Also given: Surface Area

$$S = (area\ of\ base) + (area\ of\ four\ sides)$$
$$S = 108 = \underline{x^2 + 4xh}$$

solving for h:

$$h = \frac{108 - x^2}{4x} = \frac{27}{x} - \frac{x}{4}$$

$$V(x) = (x^2)\left(\frac{27}{x} - \frac{x}{4}\right)$$

step 3 — $V(x) = 27x - \frac{x^3}{4}$

$$V'(x) = 27 - \frac{3x^2}{4}$$

$$0 = 27 - \frac{3x^2}{4}$$

$$x^2 = \frac{4}{3}(27) = 36$$

$$x \pm 6$$

Step 4 feasible solution:

$$0 < x < \sqrt{108}$$

CP: $x = 6$

x	0	6	$\sqrt{108}$
$f(x)$	0	108	0

The box needs to be 6" × 6" × 3" in to maximize Volume

We are trying to maximize volume, so...

3 → Write the volume as a function of one variable (x)

3½ → To maximize V, find the critical numbers of the volume function - take the derivative and set it equal to 0

4 → determine the interval/domain over which there may be a solution for x

Finding the Minimum Distance

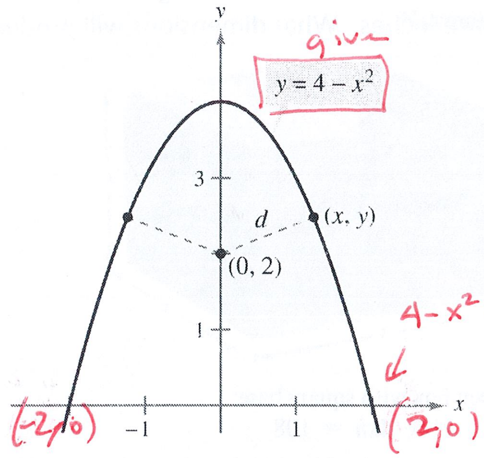
Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

→ Use the distance formula:
$d = \sqrt{(x-0)^2 + (y-2)^2}$

→ because d is the smallest when the expression inside the radical is the smallest, you need to find the critical numbers of the radicand (f')

or just use

$f(x,y) = d^2 = x^2 + (y-2)^2$



The quantity to be minimized is distance:
$d = \sqrt{(x-0)^2 + (y-2)^2}$.

$f(x) = x^2 + ((4-x^2)-2)^2$

$f(x) = x^2 + (2-x^2)^2$

$f(x) = x^2 + (4 - 4x^2 + x^4)$

$f(x) = x^4 - 3x^2 + 4$

$f'(x) = 4x^3 - 6x$

$0 = 2x(2x^2 - 3)$

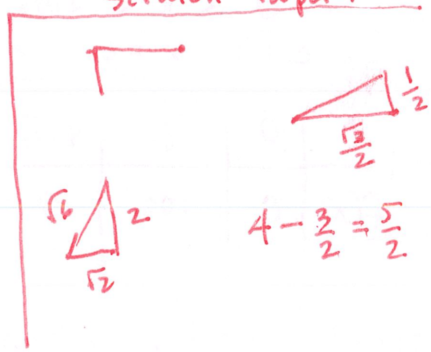
CP: $0, -\sqrt{\frac{3}{2}}, +\sqrt{\frac{3}{2}} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$

feasible Solutions

$-2 < x < 2$

f': $[-2$ ⊖ $-\sqrt{3/2}$ ⊕ 0 ⊖ $\sqrt{3/2}$ ⊕ $2]$

Scratch Paper:



$4 - \frac{3}{2} = \frac{5}{2}$

$\pm\sqrt{\frac{3}{2}}$ are rel min b/c f' Δ's from neg to pos

(or)

x	-2	$-\frac{\sqrt{6}}{2}$	0	$\pm\frac{\sqrt{6}}{2}$	2
f(x)	$2\sqrt{2}$	$\frac{5}{2}$	2	$\frac{5}{2}$	$2\sqrt{2}$

The closest points on the curve to $(0,2)$ are $\left(-\frac{\sqrt{6}}{2}, \frac{5}{2}\right)$ & $\left(\frac{\sqrt{6}}{2}, \frac{5}{2}\right)$

Finding Minimum Area

A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be 1.5 inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?

There are 2 things going on here:

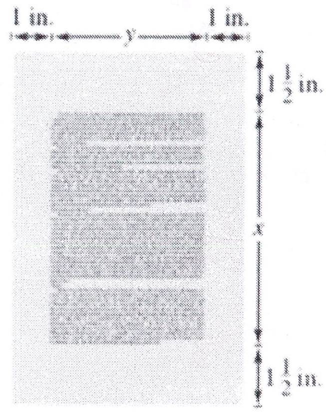
→ The area to be minimized:

$$A(x,y) = (x+3)(y+2)$$

→ The printed area inside the margins:

$$xy = 24$$

- solve for y and substitute into the primary equation
- As area is positive, we need to calculate the critical numbers, differentiate with respect to x.
- domain $x > 0$



The quantity to be minimized is area:
$A = (x + 3)(y + 2)$.

$$A(x) = (x+3)\left[\left(\frac{24}{x}\right)+2\right]$$

$$A(x) = 24 + 2x + \frac{72}{x} + 6$$

$$A(x) = 30 + 2x + \frac{72}{x}$$

$$A'(x) = 2 - \frac{72}{x^2}$$

$$A''(x) = \frac{+144}{x^2}$$

$$0 = 2 - \frac{72}{x^2}$$

$$x^2 = 36$$

CP: $x = \pm 6$

Scratch Paper

$$\boxed{\begin{array}{l} 6+3 = 9'' \\ \frac{24}{6} + 2 = 6'' \end{array}}$$

feasible solutions:

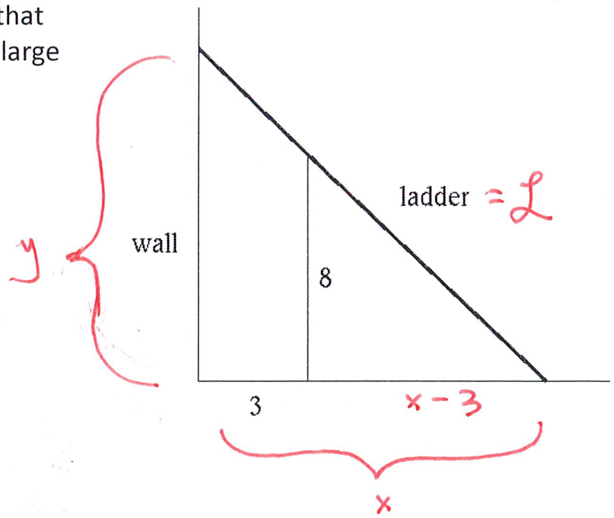
$$0 < x < \infty$$

f': neg on $(0, 6)$, pos after 6

$f'(6) = 0$, $A''(6) > 0$ concave up so $f(6)$ is a rel min by 2nd derivative Test

The minimum paper dimension w/ these margins are 6" by 9"

Find the length of the shortest ladder that will reach over an 8-ft. high fence to a large wall, which is 3 ft. behind the fence.



Minimize $\mathcal{L}$ (or $\mathcal{L}^2$)

Primary equation:

$$\mathcal{L}^2 = f(x,y) = x^2 + y^2$$

Secondary equation:

Similar Δ's

$$\frac{y}{x} = \frac{8}{x-3}$$

$$y = \frac{8x}{x-3}$$

feasible solutions:
$x > 3$

$$f(x) = x^2 + \left(\frac{8x}{x-3}\right)^2 = x^2 + \frac{64x^2}{(x-3)^2}$$

$$f'(x) = 2x + \frac{(x-3)^2(128x) - 2(x-3)\,64x^2}{(x-3)^4}$$

$$0 = 2x(x-3)^4 + 128x(x-3)^2 - 128x^2(x-3)$$

$$0 = (x-3)^3 + 64(x-3) - 64x$$

with TI-84: $x \approx 8.7889604$ $\boxed{\rightarrow}$ $\boxed{\text{ALPHA}}$ A

f' ⊖ | ⊕
A

$f'(x)$ changes from neg to pos at $x = A$, so a local min, > 3.

$$\mathcal{L} = \sqrt{f(A)} = \sqrt{A^2 + \frac{8A}{A-3}} \approx 14.9922253$$

The shortest ladder that will reach is about 15 feet long (I am not sure anyone would sell me one that is 14′ 11 29/32″ long!)

An Endpoint Maximum

Four feet of wire is to be used to form a square and a circle. How much of the wire should be used for the circle to enclose the maximum total area?

→ Total area is given by:

$$A = (area\ of\ square) + (area\ of\ circle)$$

$$A = x^2 + \pi r^2$$

→ Areas are formed by a 4 ft. length of wire, so we define the perimeter:

$$4 = (P\ of\ square) + (P\ of\ circle)$$

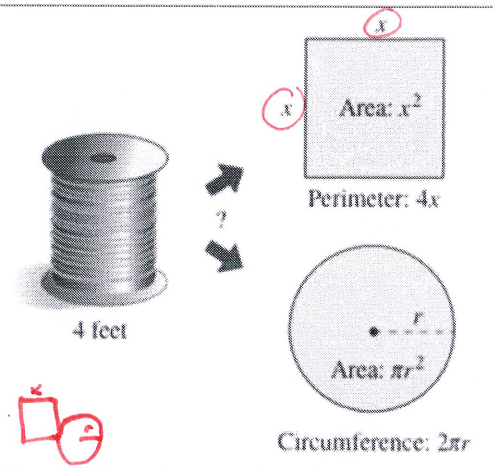
$$4 = 4x + 2\pi r$$

- Solve for r, substitute into the primary equation
- The domain of x is restricted by the fact that first, the length of the sides of the square must be greater than 0. Second, as we are limited to 4 ft. of wire and we have four sides of a square, so x can be no greater than 1.

$$D: 0 \le x \le 1$$

→ Differentiate A with respect to x

→ Find the critical number(s) in the domain



The quantity to be maximized is area: $A = x^2 + \pi r^2$.

$P = 4 = 4x + 2\pi r$

$r = \frac{4-4x}{2\pi} = \frac{2-2x}{\pi}$

4 sides of a square

feasible solutions $0 < x < 1$

Maximize Area

$$A(x) = x^2 + \pi\left(\frac{2-2x}{\pi}\right)^2 = x^2 + \frac{(4 - 8x + 4x^2)}{\pi}$$

$$A(x) = x^2 + \frac{4}{\pi} - \frac{8x}{\pi} + \frac{4}{\pi}x^2$$

$$A'(x) = 2x - \frac{8}{\pi} + \frac{8}{\pi}x$$

$$0 = x\left(2 + \frac{8}{\pi}\right) - \frac{8}{\pi}$$

$$\frac{8}{\pi} = x\left(2 + \frac{8}{\pi}\right)$$

$$x = \frac{8}{\pi} \cdot \frac{1}{2 + \frac{8}{\pi}} \approx .560$$

A' ⊖ | ⊕ at $.560$ a min!

so Candidates Test:

x	0	.560	1
$A(x)$	1.273	.560	1

Ab max

Max area is at $x = 0$ so No square, only the Circle